Modular symbols

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Modular symbols span a vector space closely related to a space of modular forms, on which the action of the Hecke algebra can be described explicitly. This makes them useful for computing with spaces of modular forms.

In this talk, we will first see that the modular symbols are nothing but homology classes, relative to the cusps, of smooth paths from cusps in the extended upper half plan $\mathcal{H}^*$, and thus they span an abelian group which is a slight enlargement of the usual homology group. Then we will make a mild generalisation, by showing a real (and complex) duality between the space $\Omega^1(X_\Gamma)$ of differential 1-forms on the modular curve X_Γ (and thus the space of weight 2 cusp forms) and the real coefficients homology group.

Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index and $\mathcal{H}^* = \mathcal{H} \cup \mathbb{QP}^1 = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. Recall that $SL_2(\mathbb{Z})$ acts on $\mathbb{QP}^1$ as follow : for $g \in SL_2(\mathbb{Z})$, $x \in \mathbb{QP}^1$

$$\begin{cases} g.x = \frac{ax+b}{cx+d} & cx+d \neq 0, x \in \mathbb{Q} \\ g.x = \infty & cx+d = 0, x \in \mathbb{Q} \\ g.\infty = \frac{a}{c} & \end{cases}$$

The action is transitive as the orbit of $\{\infty\}$ is all $\mathbb{QP}^1$. In fact for $\frac{a}{c} \in \mathbb{QP}^1, (a,c) = 1$ one has by Bezout's identity

$$\exists b, d \in \mathbb{Z} \text{ such that } ad - bc = 1 \Rightarrow \exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \, , \, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}.$$

Recall also that ${}_\Gamma\backslash^{\mathcal{H}^*}$ is a compact Hausdorff space, one can get a complex analytic structure on ${}_\Gamma\backslash^{\mathcal{H}^*} =: X_\Gamma$ as a Riemann surface. Let $\pi_\Gamma : \mathcal{H}^* \twoheadrightarrow X_\Gamma$ be the natural projection.

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1 The fundamental group and the first degree Homology

Definition 1.1. *Let $\alpha, \beta \in \mathcal{H}^*$ we define a path γ from α to β to be a piecewise smooth path in $\mathcal{H}$*

$$\begin{aligned} \gamma\colon [0,1] &\to \mathcal{H}^* \\ 0 &\mapsto \alpha \\ 1 &\mapsto \beta \end{aligned}$$

where γ is analytic at the end points : If the endpoints are cusps, say $\beta = \infty$ we require the path to be contained in a vertical strip of finite width N and that the projection under $z \mapsto e^{2i\pi z/N}$ is an analytic arc going to 0 in the unit ball.

For $\alpha, \beta \in \mathcal{H}^*$, one defines under the projection π_Γ smooth paths on the surface X_Γ, we note by $X_\Gamma^{[0,1]}$ the set of all paths in X_Γ.

Proposition 1.2. *Let γ, γ' be two paths that have same endpoints. We say that γ and γ' are homotopic if one can be continuously deformed into another.*
In particular, in $\mathcal{H}^*$ all paths that have same endpoints are homotopic, and by analicity of π_Γ, their projections on X_Γ are homotopic.

We take the quotient of $X_\Gamma^{[0,1]}$ by the homotopy relation, and we look at homotopy classes of paths in X_Γ. We are interested in a particular type of paths :

Let α and β be equivalent under the action of Γ, then the projection of the path from α to $g.\alpha$ is a closed path (or a loop) in X_Γ.

Definition 1.3. *The set of homotopy classes of loops (with base point $\pi_\Gamma(\alpha)$) in X_Γ along with a suitable product form a group, called the* ***Fundamental group*** *and is denoted $\pi_1(X_\Gamma, \pi_\Gamma(\alpha))$.*
Note that for path-connected spaces, the fundamental group is independent of the choice of the base point, so we will only write $\pi_1(X_\Gamma)$.

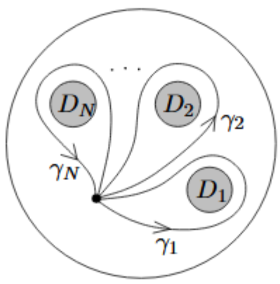

Figure 1: Fundamental group of a sphere with N holes.

Definition 1.4. *We define the* ***first integral homology group*** *of X_Γ to be the abelianization of its fundamental group :*

$$H_1(X_\Gamma, \mathbb{Z}) := \pi_1(X_\Gamma)^{Ab} = \pi_1(X_\Gamma) \Big/ [\pi_1(X_\Gamma), \pi_1(X_\Gamma)]$$

Recall that X_Γ is a Riemann surface of genus g. Topologically, it is a g-holed torus, each one has two generating loops. Thus $H_1(X_\Gamma, \mathbb{Z})$ is a free abelian group of order $2g$.

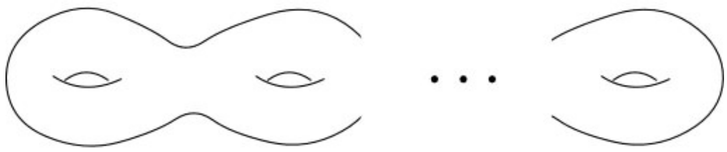

Figure 2: $H_1(X_\Gamma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$

First, let α and β be two points equivalent under the action of Γ (e.g $\alpha, \beta \in \mathbb{QP}^1$). Then any path γ from α to β in $\mathcal{H}^*$ determines an integral homology class $\overline{\gamma}$ in $H_1(X_\Gamma, \mathbb{Z})$, which depends only on α and β and not on the path chosen, as $\mathcal{H}^*$ is simply connected. We will denote each class by $\{\alpha, \beta\}_\Gamma$ and call it a **Modular symbol**. We will omit the index of the group when it is clear from context.

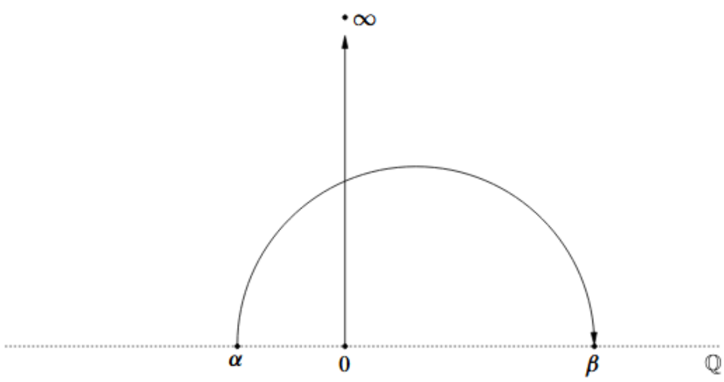

Figure 3: The modular symbols $\{\alpha, \beta\}_\Gamma$ and $\{0, \infty\}_\Gamma$

2 Duality

Let $\Omega^1(X_\Gamma)$ be the g-dimensional $\mathbb{C}$-vector space of differential 1-forms on the Riemann surface X_Γ.
Every integral homology class $\gamma \in H_1(X_\Gamma, \mathbb{Z})$ can be represented by a modular symbol $\{\alpha, \beta\}$ in the following way : Every homology class $\overline{\gamma}$ in $H_1(X_\Gamma, \mathbb{Z})$ gives a homotopy class $[\gamma]$ in $\mathcal{H}^*$. Now one has to be slightly careful, consider $\mathcal{H}^0$ to be $\mathcal{H} \setminus \Gamma i \cup \Gamma\rho$ and X_Γ^0 its projection. Then one sees that π_Γ^0 is an unramified covering and thus $\mathcal{H}^0$ is the universal cover of X_Γ^0

$$\begin{tikzcd}
 & \mathcal{H}^0 \arrow[r, hook] \arrow[d, "\pi_\Gamma^0"] & \mathcal{H}^* \arrow[d, "\pi_\Gamma"] \\
I = [0,1] \arrow[r, "\gamma"] \arrow[ur, dashed, "\widehat{\gamma}"] & X_\Gamma^0 \arrow[r, hook] & X_\Gamma
\end{tikzcd}$$

Where $\widehat{\gamma}(0) := \alpha$ and $\widehat{\gamma}(1) := \beta$ are in the same fiber, i.e are Γ-equivalent. For the other points, one works it out locally.
Now for $\omega \in \Omega^1(X_\Gamma)$

$$\int_{\overline{\gamma}} \omega = \int_{\gamma} \pi^* \omega = \int_{\alpha}^{\beta} \omega \tag{1}$$

is well defined, since integrating over a 'homology classe' does not depend on the path γ (One calls this special kind of linear functional a *period*).
Recall that

$$H_1(X_\Gamma, \mathbb{R}) \simeq H_1(X_\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

is a $2g$-dimensional $\mathbb{R}$-vector space obtained by formal extension of scalars from $H_1(X_\Gamma, \mathbb{Z})$ (Explicitly, and for every ring R one just takes R-linear combinations of the $2g$ generators of the $\mathbb{Z}$-module $H_1(X_\Gamma, \mathbb{Z})$, the result is then an R-module).

Theorem 2.1. *The first homolgy group $H_1(X_\Gamma, \mathbb{R})$ with real coefficients is dual to $\Omega^1(X_\Gamma)$ as $\mathbb{R}$ vector spaces.*

Proof. Let $\{\overline{\gamma}_j\}_{0 \leq j \leq 2g}$ be a fixed homology basis i.e. a $\mathbb{Z}$-basis of $H_1(X_\Gamma, \mathbb{Z})$ and $\{\omega_k\}_{0 \leq k \leq g}$ a fixed basis of $\Omega^1(X_\Gamma)$.

- We may identify $H_1(X_\Gamma, \mathbb{R})$ with the space of all $\mathbb{C}$-linear functionals on $\Omega^1(X_\Gamma)$ as follows: given an element $\overline{\gamma} \in H_1(X_\Gamma, \mathbb{R})$, we can write $\overline{\gamma}$ uniquely in the form

  $$\overline{\gamma} = \sum_{j=1}^{2g} c_j \overline{\gamma}_j^{\mathbb{Z}}$$

  with coefficients $c_j \in \mathbb{R}$. As every integral homology class gives a $\mathbb{C}$-linear functional $\omega \mapsto \int_{\overline{\gamma}} \omega$, we define

  $$\langle \overline{\gamma}, \omega \rangle = \sum_{j=1}^{2g} c_j \int_{\overline{\gamma}_j} \omega_k$$

  Then $\overline{\gamma}$ corresponds to a $\mathbb{C}$-linear functional $\omega \mapsto \langle \overline{\gamma}, \omega \rangle$.

– Conversely, given a functional $f : \Omega^1(X_\Gamma) \to \mathbb{C}$, we first consider the $2g \times g$ complex *period matrix*

$$\Omega := (w_{jk}) = (\int_{\overline{\gamma}_j} \omega_k) \tag{2}$$

By Riemann's bilinear identities (see ref [2]) one sees that the row vectors

$$\Omega_j = (\int_{\overline{\gamma}_j} \omega_1, \; \dots \;, \int_{\overline{\gamma}_j} \omega_k)$$

are $\mathbb{R}$-linearly independents, and so their $\mathbb{Z}$-span is a lattice Λ of rank $2g$ in $\mathbb{C}^g$. One defines the *Jacobian* of X_Γ to be

$$J(X_\Gamma) = \mathbb{C}^g \,/\Lambda$$

Thus the vector $(f(\omega_1),...,f(\omega_g))$ in $\mathbb{C}^g$ may be expressed uniquely as an $\mathbb{R}$-linear combination of the rows Ω_j

$$f(\omega_i) = \sum_{j=1}^{2g} c_j \int_{\overline{\gamma}_j} \omega_i$$

for all $\omega \in \Omega^1(X_\Gamma)$. Hence $f(\omega) = \langle \overline{\gamma}, \omega \rangle$, where $\overline{\gamma} = \sum_{j=1}^{2g} c_j \overline{\gamma}_j^{\mathbb{Z}}$.

□

In particular, let $\alpha, \beta \in \mathcal{H}^*$ be arbitrary (not necessarily in the same Γ-orbit) then the functional $\omega \mapsto \int_\alpha^\beta \omega$ corresponds to a unique element $\overline{\gamma} \in H_1(X_\Gamma, \mathbb{R})$ where

$$\overline{\gamma} = \sum_{j=1}^{2g} c_j \overline{\gamma}_j^{\mathbb{Z}}$$

We extend thus our definition of modular symbol to be that element, namely

$$\{\alpha, \beta\}_\Gamma = \overline{\gamma} = \sum_{j=1}^{2g} c_j \overline{\gamma}_j^{\mathbb{Z}} \in H_1(X_\Gamma, \mathbb{R})$$

One sees that if $c_j \in \mathbb{Z}$ then this definition agrees with the one we had earlier.
We get then an $\mathbb{R}$-biliniear pairing.

$$\begin{aligned} H_1(X_\Gamma, \mathbb{R}) \times \Omega^1(X_\Gamma) &\longrightarrow \mathbb{C} \\ (\{\alpha, \beta\}_\Gamma, \omega) &\longmapsto \langle \{\alpha, \beta\}_\Gamma, \omega \rangle = \langle \overline{\gamma}, \omega \rangle = \int_{\overline{\gamma}} \omega = \int_\alpha^\beta \omega \end{aligned} \tag{3}$$

which gives an exact duality between the two spaces on the left if we view $\Omega^1(X_\Gamma)$ as a real vector space of dimension $2g$ by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$.

3 Modular symbols formalism

Motivated by this picture, we declare that the modular symbols satisfy the following formalism, mainly homology and quotient by torsion relations :

Proposition 3.1. *Let* $\alpha, \beta, \delta \in \mathcal{H}^*$

1. $\{\alpha, \alpha\} = 0$ *and* $\{\alpha, \beta\} + \{\beta, \alpha\} = 0$
2. $\{\alpha, \beta\} + \{\beta, \delta\} = \{\alpha, \delta\}$
3. *For* $g \in GL_2(\mathbb{Q})^+$, $\omega \in \Omega^1(X_\Gamma)$
$$\langle \{g\alpha, g\beta\}, \omega \rangle = \langle \{\alpha, \beta\}, g.\omega \rangle$$
4. *For* $g \in \Gamma$
$$\{g\alpha, g\beta\}_\Gamma = \{\alpha, \beta\}_\Gamma$$

Proof.
1. $\{\alpha, \alpha\} = 0$ since a loop in α is contractible.
$\{\alpha, \beta\} + \{\beta, \alpha\} = 0$ is also clear from Cauchy's theorem.
2. $\{\alpha, \beta\} + \{\beta, \delta\} - \{\alpha, \delta\} = \{\alpha, \beta\} + \{\beta, \delta\} + \{\delta, \alpha\} = 0$ by the same argument as in (1).
3. Let γ be a path from α to β, then $g\gamma$ is a path from $g\alpha$ to $g\beta$. By a change of variable we get
$$\langle \{\alpha, \beta\}, g.\omega \rangle = \int_\alpha^\beta g.\omega = \int_\gamma g.\omega = \int_{g.\gamma} \omega = \int_{g.\alpha}^{g.\beta} \omega = \langle \{g\alpha, g\beta\}, \omega \rangle$$
4. As $g \in \Gamma$ one has that $g.\gamma \in \overline{\gamma}$ in $H_1(X_\Gamma, \mathbb{R})$ thus the equality.

$\square$

Moreover, one deduces easily the following properties from 3.1 :

(i) For $g \in \Gamma$
$$\{\alpha, g\alpha\}_\Gamma = \{\beta, g\beta\}_\Gamma$$
Indeed, $\{\alpha, g\alpha\}_\Gamma = \{\alpha, \beta\}_\Gamma + \{\beta, g\beta\}_\Gamma + \{g\beta, g\alpha\}_\Gamma = \{\alpha, \beta\}_\Gamma + \{\beta, g\beta\}_\Gamma + \{\beta, \alpha\}_\Gamma = \{\beta, g\beta\}_\Gamma$

(ii) For $g_1, g_2 \in \Gamma$
$$\{\alpha, g_1 g_2 \alpha\}_\Gamma = \{\alpha, g_1\alpha\}_\Gamma + \{\alpha, g_2\alpha\}_\Gamma$$
indeed, $\{\alpha, g_1 g_2 \alpha\}_\Gamma = \{\alpha, g_1\alpha\}_\Gamma + \{g_1\alpha, g_1 g_2\alpha\}_\Gamma = \{\alpha, g_1\alpha\}_\Gamma + \{\alpha, g_2\alpha\}_\Gamma$

And thus one gets :

Corollary 3.2. *The map*
$$\begin{aligned} \Gamma &\longrightarrow H_1(X_\Gamma, \mathbb{Z}) \\ g &\longmapsto \{\alpha, g\alpha\} \end{aligned}$$
is a surjective group homomorphism, that does not depend on α.

References

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[3] J.I. Manin *Parabolic points and zeta functions of modular curves* Izv. Akad. Nauk SSSR. Ser. Mat. **36** (1972), 19-66. MR47